

On Nonblocking Folded-Clos Networks in Computer Communication Environments

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Abstract—Folded-Clos networks, also referred to as fat-trees, have been widely used as interconnects in large scale high performance computing clusters. The switching capability of such interconnects in computer communication environments, however, is not well understood. In particular, the concept of nonblocking interconnects, which is often used by system vendors, has only been studied in the telephone communication environment with the assumption of a centralized controller. Such “nonblocking” networks do not support nonblocking communications in computer communication environments where the network control is distributed. This paper theoretically analyzes the conditions for folded-Clos networks to achieve nonblocking communications in computer communication environments with various routing schemes including deterministic routing and adaptive routing, and establishes nonblocking conditions.

Keywords—Clos Networks; fat-tree; nonblocking; computer communications;

I. INTRODUCTION

Clos networks and their variations such as folded-Clos (also referred to as fat-trees) have been widely used for multiprocessor interconnects and system area networks. Almost all large scale commodity high performance computing clusters are interconnected with such topologies.

A three-stage Clos network has the input stage, the middle stage, and the output stage. The input stage consists of $n \times m$ switches; the middle stage consists of $r \times r$ switches; and the output stage consists of $m \times n$ switches. There are r input switches, r output switches, and m middle switches with each of the input and output switches having a link connecting to each of the middle switches. Fig. 1 (a) depicts a three-stage Clos network. We will use the notion $Clos(n, m, r)$ to denote the Clos network with parameters n , m , and r . Notice that the links in $Clos(n, m, r)$ are uni-directional. A folded-Clos (fat-tree) network is the one-sided version of the Clos network: it basically merges the corresponding input and output switches into one switch. Fig. 1 (b) shows a folded-Clos (fat-tree) network. Each edge in a folded-Clos network consists of two directed links; lower level switches are $(n + m) \times (n + m)$ switches; and top level switches are $r \times r$ switches. We will use the notion $ftree(n, m, r)$ to denote such a fat-tree, which is logically equivalent to $Clos(n, m, r)$.

The switching capability of $ftree(n, m, r)$ (or $Clos(n, m, r)$) is determined by the parameters n , m , and r . The study of such networks has focused on finding the most cost-effective values for n , m , and r to achieve

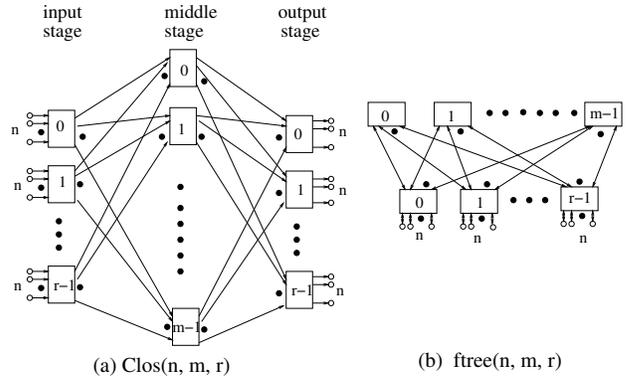


Fig. 1. Clos and folded-Clos networks

nonblocking communication, that is, the ability to establish a connection from an arbitrary input port to an arbitrary output port without causing contention. A nonblocking network can provide connectivity for any permutation communication, which consists connections from arbitrary input ports to arbitrary output ports with the restriction that each input or output port can be used at most once in the communication. The most important results in this area are summarized in various nonblocking conditions for Clos networks including *strictly nonblocking* [2], *wide-sense nonblocking* [4], [16], and *rearrangeably nonblocking* [3]. Although these nonblocking conditions results are significant, all of them were obtained with the assumption that a centralized controller is used to manage all network resources, which renders the results not applicable to computer communications where the network control is distributed.

When used in computing clusters, folded-Clos based interconnects are often treated as the replacement of central crossbar switches that support any permutation communication with full bisection bandwidth [7]. Although many folded-Clos based interconnects for clusters are “nonblocking” (strictly, wide sense, or rearrangeably) in theory, the delivered performance is far-from that of crossbar switches [5], [7]. Without the centralized controller that takes all connection requests and manages all network resources, even a strictly nonblocking network can still block a permutation communication.

We define a *nonblocking folded-Clos network in the computer communication environment* to be one that, with dis-

tributed control, can support any permutation communication without network contention. Using this definition, if a folded-Clos based interconnect is nonblocking, it can support any permutation communication with no contention and can thus achieve full bisection bandwidth for any permutation communication: such an interconnect behaves like a crossbar switch.

The understanding of folded-Clos based interconnects in computer communication environments is insufficient: techniques for building truly nonblocking folded-Clos interconnects in computer communication environments have not been developed; it is unclear what the cost of such interconnects will be. Through theoretical analysis, this paper gives answers to these important questions. We investigate folded-Clos networks that are nonblocking in computer communication environments, develop techniques for building such networks, and establish nonblocking conditions for folded-Clos networks with distributed control under various routing schemes including deterministic routing and adaptive routing. The major results include the following, which can be used in feasibility analysis for building nonblocking folded-Clos based interconnects under other constraints, and directly applied to build such interconnects.

- With single-path deterministic routing, it is not cost-effective to build nonblocking $ftree(n, m, r)$ with $r \leq 2n + 1$. Using relatively small top level switches to build nonblocking $ftree(n, m, r)$ is not effective.
- With single-path deterministic routing, when $r \geq 2n + 1$, the nonblocking condition for $ftree(n, m, r)$ is $m \geq n^2$. Using single-path deterministic routing, $O(N\sqrt{N})$ -port nonblocking interconnects can be constructed using $O(N)$ N -port switches. Commonly used multi-path deterministic routing schemes have the same nonblocking condition as single-path routing.
- With local adaptive routing where routes may adapt based on the information local to each switch, when $r \leq n^c$, where c is a constant, there exists a function $f(n) = O(n^{2-\frac{1}{2(c+1)}})$ such that the nonblocking condition for $ftree(n, m, r)$ is $m \geq f(n)$. Local adaptive routing allows using a smaller number of top level switches to achieve nonblocking communication in comparison to deterministic routing.

The rest of the paper is structured as follows. Section II discusses the related work. Section III describes the background and notations used in the paper. Section IV considers deterministic routing, derives the nonblocking condition, and presents the routing scheme for nonblocking $ftree(n, m, r)$ with single-path deterministic routing. Section V considers adaptive routing and shows that local adaptive routing improves the nonblocking condition over deterministic routing. Finally, Section VI concludes the paper.

II. RELATED WORK

The switching capability of Clos networks in the context of switching telephone traffics has been extensively studied. Various nonblocking conditions have been established. A Clos network is *strictly nonblocking* if it is always possible to

set up a connecting path from an idle input port to an idle output port independent of the existing connections and the path search algorithm. Clos showed [2] that $Clos(n, m, r)$ is strictly nonblocking if the number of middle stage switches $m \geq 2n - 1$. A network is *wide-sense nonblocking* when it is always possible to set up a path from an idle input port to an idle output by suitably choosing routes for new connections. The conditions for a network to be wide-sense nonblocking depend on the routing algorithm; some results are obtained by Benes [4] and Yang [16]. A network is *rearrangeably nonblocking* if a new connection from an idle input port to an idle output port can always be established by rearranging the paths for existing connections. Benes [3] showed that a Clos network is rearrangeably nonblocking if $m \geq n$. All these results assume a centralized controller and cannot be directly applied to computer networks with distributed control.

After Leiserson introduced the fat-tree topology to computer networks [10], [11], the topology has become popular and extensive research has been performed on this topology. The topology has been greatly extended [13], [14]. Since traditional nonblocking networks are blocking in the computer communication environment, most studies have focused on understanding the performance of such networks by analyzing the blocking probability [6], [9], [15], or developing techniques to reduce the blocking probability [1], [6], [9], [15], [17]. Various routing techniques including randomized routing [6], [15], multi-path routing [1], [17], and adaptive routing [9], have been proposed to minimize the blocking probability. Even with all these improvements, recent studies still show that the contemporary fat-tree based interconnects offer much lower performance than crossbar switches [5], [7]. This paper investigates the conditions for folded-Clos networks to be nonblocking in computer communication environments, which differs from all existing work in the area.

III. BACKGROUND AND NOTATIONS

This paper focuses on $ftree(n, m, r)$'s that are nonblocking in computer communication environments. An example $ftree(n, m, r)$ is shown in Fig. 1 (b). There are two layers of switches and one layer of leaf nodes in the topology. The leaf nodes are communication sources and destinations. $ftree(n, m, r)$ has r bottom level n, m -port switches and m top level r -port switches. It supports $r \times n$ leaf nodes. Each edge in the topology consists of two directed links. The links from leaf nodes to bottom level switches and from bottom level switches to top level switches are *uplinks*. The links from top level switches to bottom level switches and from bottom level switches to leaf nodes are *downlinks*. As shown in Fig. 1 (b), we number the m top level switches from 0 to $m - 1$, the r bottom level switches from 0 to $r - 1$, and the $r \times n$ leaf nodes from 0 to $r \times n - 1$. Other numbering schemes will also be used in the paper. They will be introduced before they are used.

Let us denote (s, d) a source-destination (SD) pair with source node s to destination node d . $SRC(s, d)$ is the switch that s is in and $DST(s, d)$ is the switch d is in. We say

that $SRC(s, d)$ is the source switch of (s, d) and that (s, d) starts from $SRC(s, d)$. Similarly, $DST(s, d)$ is the destination switch of (s, d) and (s, d) ends at $DST(s, d)$. We will use the phrase *SD pairs from the same switch* to denote SD pairs whose sources are in the same switch, *SD pairs from different switches* to denote SD pairs whose sources are in different switches, *SD pairs to the same switch* to denote SD pairs whose destinations are in the same switch, and *SD pairs to different switches* to denote SD pairs whose destinations are in different switches. A communication pattern can be represented by a set of SD pairs.

Definition 1: A *permutation communication*, or permutation, is a communication pattern where each leaf node can be the source in at most one SD pair and the destination in at most one SD pair in the communication pattern.

Property 1: Let (s_1, d_1) and (s_2, d_2) be two SD pairs in a permutation. $s_1 \neq s_2$ and $d_1 \neq d_2$.

This property can be obtained from the definition of permutation. Since $ftree(n, m, r)$ have $r \times n$ leaf nodes, a permutation communication can at most have $r \times n$ SD pairs. When all source nodes and all destination nodes are used in a permutation, there are exactly $r \times n$ SD pairs. A permutation, however, does not require all leaf nodes to be used.

To support the communication for a SD pair, a path must be used to carry the traffics for the communication. In computer communication environments, distributed control is performed by the routing algorithm, which determines the path for each packet. We consider several widely used routing algorithms for folded-Clos networks: single-path deterministic routing, multi-path deterministic routing, and adaptive routing. In single-path deterministic routing, one path is used to carry all traffics for each SD pair and the path for each SD pair is pre-determined. In (traffic oblivious) multi-path deterministic routing, the traffics for the same SD pair are distributed among multiple pre-determined paths either in a deterministic or random manner. For both single path deterministic routing and multi-path deterministic routing, the paths used are independent of the traffic pattern. In adaptive routing, different paths can be used for one SD pair and the path to be used is determined dynamically based on the traffic condition.

Some adaptive routing algorithms require routes to adapt based on the whole communication pattern. We call such algorithms *global adaptive routing algorithms* since the whole communication pattern must be considered in order to determine a path. Global adaptive routing is equivalent to routing with a centralized controller (so that the whole communication pattern is known to the routing algorithm); and the nonblocking conditions for such routing schemes on fat-trees have been established. In this paper, we focus on *local adaptive routing* where routes adapt only based on the information that is available locally to each switch.

With a given routing algorithm, when packets in two SD pairs in a communication pattern are routed through one network link, we say that the communication pattern causes network *contention*.

Definition 2: A folded-Clos network is *nonblocking with*

a routing algorithm if any permutation communication can be supported without network contention using the routing algorithm on the network.

IV. DETERMINISTIC ROUTING

This section considers nonblocking folded-Clos networks with deterministic routing. We will first consider single-path deterministic routing and then discuss multi-path routing.

A. Single-path deterministic routing

In single-path deterministic routing, a path is deterministically assigned to each SD pair. The following lemma gives the condition for a folded-Clos network to be nonblocking with single-path deterministic routing.

Lemma 1: For any single-path deterministic routing, $ftree(n, m, r)$ is nonblocking if and only if each directed link carries traffics either from one source or to one destination.

Proof: We will prove the necessary condition by contradiction. Let a directed link L in $ftree(n, m, r)$ carries traffics from more than one source and to more than one destination. There exists at least two SD pairs, (s_1, d_1) and (s_2, d_2) , $s_1 \neq s_2$ and $d_1 \neq d_2$, whose traffics are routed through L . By Definition 1, the communication pattern that contains only these two SD pairs $\{(s_1, d_1), (s_2, d_2)\}$ is a permutation. Since the routing is deterministic, there is contention on link L for this permutation and thus, the network is blocking. Hence, if $ftree(n, m, r)$ is nonblocking, each link in the network carries traffics either from one source or one destination.

We will now prove the sufficient condition by contradiction. Assume that we have a permutation P that can cause network contention: there exists two SD pairs in P that are routed through one link. Let the two SD pairs be (s_1, d_1) and (s_2, d_2) and the link be L . Since P is a permutation, we have $s_1 \neq s_2$ and $d_1 \neq d_2$ (Property 1). This contradicts to the assumption that link L only carries traffics either from one source or to one destination. Hence, if each link in the network carries traffics either from one source or to one destination, $ftree(n, m, r)$ is nonblocking. \square

In $ftree(n, m, r)$, each link between a leaf node and a bottom level switch only connects to one leaf node: the traffics on such a link is either to that leaf node or from the leaf node regardless of the routing algorithm. Such a link does not have contention for any permutation. The links between top level switches and bottom level switches may have contention. Since a nonblocking network must support any permutation, all possible SD pairs must be assigned a path by the routing algorithm. For SD pair (s, d) , where s and d are not in the same bottom level switch, it must be routed through a top level switch and use the links between top level switches and bottom level switches. There are $r(r-1)n^2$ such SD pairs for $ftree(n, m, r)$ that must be routed carefully in order to achieve nonblocking communication.

In deriving the nonblocking condition for single-path deterministic routing, we must determine the smallest m such that all of the $r(r-1)n^2$ SD pairs can be routed with each link supporting SD pairs with either the same source or the

same destination. We will use a subgraph of $ftree(n, m, r)$ to analyze the number of SD pairs that can be routed through one top level switch. Fig. 2 shows the subgraph, which contains all lower level switches in $ftree(n, m, r)$, but only one top level switch. The subgraph is effectively $ftree(n + 1, r)$, a regular tree topology with the root having r children and each bottom level switch having n leaf nodes.

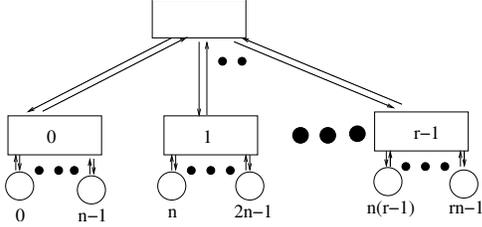


Fig. 2. The subgraph of $ftree(n, m, r)$ ($ftree(n + 1, r)$)

Lemma 2: Consider using the $ftree(n + 1, r)$ topology to route a subset of all possible SD pairs with source and destination in different switches. If each link can carry traffics either from one source or to one destination, then the largest number of SD pairs that can be routed through the root is at most $r \times (r - 1)$ when $r \geq 2n + 1$, and $2nr$ when $r \leq 2n + 1$. *Proof:* Let S be a largest set of SD pairs that are routed through the root switch when all links carry traffics either to one destination or from one source. To count the number of SD pairs in S , we partition the SD pairs in S into three types: (1) the SD pairs whose source switch has 2 or more sources in the SD pairs in S , (2) the SD pairs whose destination switch has 2 or more destinations in the SD pairs in S , and (3) the SD pairs whose source switch has one source and whose destination switch has one destination in the SD pairs in S . Let us denote the number of type (1) SD pairs in S be NUM_1 , the number of type (2) SD pairs be NUM_2 , the number of type (3) SD pairs be NUM_3 , and the total number of SD pairs in S be NUM .

Let the number of switches that have 2 or more sources in the SD pairs in S be A ; the number of switches that have 2 or more destinations be B . Consider the number of type (1) SD pairs. Since there are two or more sources in each of such switches, all these sources in one switch must communicate with one destination. Otherwise, the link from the switch to the root will carry SD pairs from more than one source and to more than one destination. Hence, one switch can contribute at most n such SD pairs to NUM_1 (when all of the leaf nodes in the switch are sources). Since there are A such switches, the number of such SD pairs, NUM_1 , is at most $A \times n$.

$$NUM_1 \leq A \times n.$$

Similarly,

$$NUM_2 \leq B \times n.$$

Now, consider the number of type (3) SD pairs, NUM_3 . Since there are A switches that cannot be source switches for type (3) SD pairs, at most $r - A$ switches can be source switches

for such SD pairs. Since each switch can at most have one destination in type (3) SD pairs, each source can at most communicate to $r - 1$ destinations (one destination in each of the $r - 1$ switches other than the source switch). Hence,

$$NUM_3 \leq (r - A) \times (r - 1).$$

Using a similar logic, by excluding switches with 2 or more destinations, we have

$$NUM_3 \leq (r - B) \times (r - 1).$$

Combine these two inequations, we obtain

$$NUM_3 \leq r \times (r - 1) - \left(\frac{A+B}{2}\right)(r - 1).$$

Therefore, the total number of SD pairs,

$$\begin{aligned} NUM &\leq NUM_1 + NUM_2 + NUM_3 \\ &\leq A \times n + B \times n + r \times (r - 1) - \frac{A+B}{2} \times (r - 1) \\ &= r \times (r - 1) + \frac{A+B}{2} \times (2 \times n + 1 - r) \end{aligned}$$

When $r \geq 2n + 1$, $\frac{A+B}{2} \times (2 \times n + 1 - r) \leq 0$ and

$$NUM \leq r \times (r - 1).$$

When $r \leq 2n + 1$, $\frac{A+B}{2} \times (2 \times n + 1 - r) \geq 0$. Since $A \leq r$ and $B \leq r$,

$$\begin{aligned} NUM &\leq r \times (r - 1) + \frac{A+B}{2} \times (2 \times n + 1 - r) \\ &\leq r \times (r - 1) + \frac{r+r}{2} \times (2 \times n + 1 - r) \\ &= 2 \times n \times r. \end{aligned}$$

□

Theorem 1: when $r \leq 2n + 1$, the number of ports supported by a nonblocking $ftree(n, m, r)$ with any single-path deterministic routing is no more than $2(n, m)$.

Proof: Regardless of the routing algorithm used, a total of $r(r - 1)n^2$ SD pairs must be routed through top level switches in $ftree(n, m, r)$. From Lemma 1 and Lemma 2, when $r \leq 2n + 1$, each top level switch can route at most $2 \times n \times r$ SD pairs in a nonblocking $ftree(n, m, r)$ for any single-path deterministic routing scheme. Hence, there are at least $\frac{r(r-1)n^2}{2nr} = \frac{(r-1)n}{2}$ top level switches needed; and $m \geq \frac{(r-1)n}{2}$. The number of ports supported by $ftree(n, m, r)$ is $r \times n \leq 2\left(\frac{r-1}{2} \times n + n\right) \leq 2(m + n)$. □

Theorem 1 indicates that it is not effective to build nonblocking folded-Clos networks using relatively small top level switches. When $r \leq 2n + 1$, the total number of ports supported by a nonblocking folded-Clos network is at most twice that in its bottom level switches. Hence, one should focus on nonblocking folded-Clos networks with relatively large top level switches ($r \geq 2n + 1$).

Theorem 2: Let $ftree(n, m, r)$ be nonblocking with any single-path deterministic routing. When $r \geq 2n + 1$, $m \geq n^2$.

Proof: Similar to the proof of Theorem 1, regardless of the routing algorithm used, $r(r - 1)n^2$ SD pairs must be routed through top level switches. From Lemma 1 and Lemma 2, when $r \geq 2n + 1$, each top level switch can route at most $r(r - 1)$ SD pairs in a nonblocking $ftree(n, m, r)$ for any single-path deterministic routing scheme. Hence, there are at

least $\frac{r(r-1)n^2}{r(r-1)} = n^2$ top level switches needed; and $m \geq n^2$.
□

Theorem 2 gives the lower bound of the number of top level switches needed to make $ftree(n, m, r)$ nonblocking with any single-path deterministic routing. The following theorem establishes that this lower bound can be achieved: the $m \geq n^2$ nonblocking condition is tight.

Theorem 3: There exists a single-path deterministic routing algorithm for $ftree(n, m = n^2, r)$ that supports all permutations without network contention. In other words, $ftree(n, m = n^2, r)$ is nonblocking using that routing algorithm.

Proof: We will first describe the routing algorithm and then prove $ftree(n, m = n^2, r)$ is nonblocking with the routing algorithm.

In $ftree(n, m = n^2, r)$, there are n^2 top level switches. We will number of n^2 top level switches by (i, j) , $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$. There are r bottom level switches numbered from 0 to $r-1$. Each bottom level switch v , $0 \leq v \leq r-1$, connects to n leaf nodes numbered as (v, k) , $0 \leq k \leq n-1$. The routing algorithm routes SD pair $(s = (v, i), d = (w, j))$, $0 \leq v \neq w \leq r-1$ and $0 \leq i, j \leq n-1$, through top level switch (i, j) . That is, SD pair $(s = (v, i), d = (w, j))$ is routed through path $(v, i) \rightarrow v \rightarrow (i, j) \rightarrow w \rightarrow (w, j)$. Note that when $v = w$, $(s = (v, i), d = (v, j))$ is routed through path $(v, i) \rightarrow v \rightarrow (v, j)$.

Using this algorithm, each uplink in $ftree(n, m = n^2, r)$ carries traffics from one source. This obviously holds for the uplinks from leaf nodes to bottom level switches. Consider the uplink from an arbitrary bottom level switch v to an arbitrary top level switch (i, j) . There are $r-1$ SD pairs on this link: $(s = (v, i), d = (0, j))$, $(s = (v, i), d = (1, j))$, ..., $(s = (v, i), d = (v-1, j))$, $(s = (v, i), d = (v+1, j))$, ..., $(s = (v, i), d = (r, j))$. There is only one source (v, i) for all the SD pairs. Similarly, each downlink in $ftree(n, m = n^2, r)$ carries traffics to one destination. Consider the downlink from an arbitrary top level switch (i, j) to an arbitrary bottom level switch v . There are $r-1$ SD pairs on this link: $(s = (0, i), d = (v, j))$, $(s = (1, i), d = (v, j))$, ..., $(s = (v-1, i), d = (v, j))$, $(s = (v+1, i), d = (v, j))$, ..., $(s = (r, i), d = (v, j))$. There is only one destination (v, j) in all the SD pairs. Fig. 3 shows the SD pairs routed through the links between top level switch (i, j) and bottom level switch v . Hence, the algorithm routes SD pairs such that each link in $ftree(n, m = n^2, r)$ carries traffics either from at most one source or to at most one destination. It follows from Lemma 1 that the network is nonblocking. □

Discussion

The main application of Clos networks is to build large (nonblocking) interconnects from smaller switches. Here, we compare nonblocking folded-Clos networks in computer communication environments with tradition rearrangeably nonblocking networks in their capability for building larger interconnects.

Consider the case when the same sized switches are used to build the nonblocking networks. That is, $r = m + n$. We

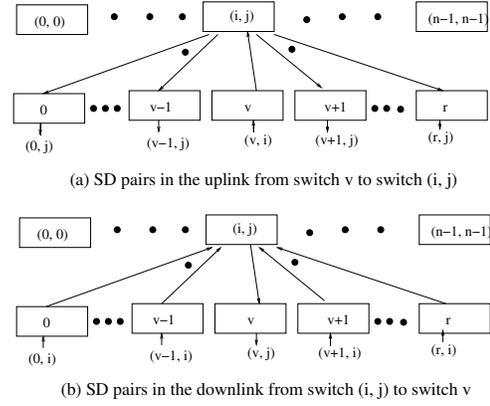


Fig. 3. SD pairs routed through links between top level switch (i, j) and bottom level switch v

will use the m -port n -trees ($FT(m, n)$) [12] as a representative rearrangeably nonblocking folded-Clos in the comparison. Our two level nonblocking folded-Clos network, $ftree(n, m = n^2, r = n + n^2)$, uses $2n^2 + n$ $n^2 + n$ -port switches to support $n^3 + n^2$ nonblocking ports. Let $N = n^2 + n$, our nonblocking network uses roughly $2N$ N -port switches to support roughly $N^{\frac{3}{2}}$ nonblocking ports. Traditional $FT(N, 2)$ uses $\frac{3N}{2}$ N -port switches to support $\frac{N^2}{2}$ ports [12].

Table I compares that number of ports and the number of switches needed for the two types of folded-Clos networks using practical building blocks: 20-port, 30-port, and 42-port switches. Our nonblocking networks behave like crossbar switches while $FT(m, n)$ is not nonblocking in computer communication environments. It is thus expected that our nonblocking networks are more expensive to construct. Our technique allows larger nonblocking folded-Clos networks to be constructed from smaller switches as shown in Table I, and is optimal for building such networks with single-path deterministic routing.

To support larger numbers of ports, the method to build 2-level nonblocking folded-Clos networks can be recursively applied to build more levels of nonblocking folded-Clos networks. For example, to obtain a 3-level nonblocking network, a 2-level nonblocking network can be used to replace each of either the top level switches or the bottom level switches in 2-level networks. Since our nonblocking $ftree(n, m, r)$ supports all permutations with no contention, it can be shown by induc-

| building block size | $ftree(n, m = n^2, r = n + n^2)$ | | $FT(n + n^2, 2)$ | |
|-----------------------|----------------------------------|------------|------------------|------------|
| | # of switches | # of ports | # of switches | # of ports |
| 20-port ($4 + 4^2$) | 36 | 80 | 30 | 200 |
| 30-port ($5 + 5^2$) | 55 | 150 | 45 | 450 |
| 42-port ($6 + 6^2$) | 88 | 252 | 63 | 884 |

TABLE I
SIZE OF NONBLOCKING $ftree(n, m = n^2, r = n + n^2)$ AND $FT(n + n^2, 2)$

tion that the recursively-built larger network will also support all permutations with no contention and is thus nonblocking. One question is whether it is more effective to replace bottom level switches or top level switches with a fat-tree. Theorem 1 gives the answer to this question: it is more beneficial to have large top level switches. Hence, when building more than two levels folded-Clos networks, one should replace top level switches with nonblocking networks. Using this approach, a three-level nonblocking folded-Clos network built with $n+n^2$ -port switches will resemble $ftree(n, m = n^2, r = n^3 + n^2)$, with each top level $n^3 + n^2$ -port switch being realized with a $ftree(n, m = n^2, r = n + n^2)$. This nonblocking network has $2n^4 + 3n^3 + n^2$ $n^2 + n$ -port switches and supports $n^4 + n^3$ ports. Let $N = n + n^2$. The three-level nonblocking folded-Clos network uses $O(N^2)$ $O(N)$ -port switches to obtain an $O(N^2)$ -port nonblocking network. As a comparison, $FT(N, 3)$ uses $O(N^2)$ $O(N)$ -port switches to obtain $O(N^3)$ -port fat-trees.

B. Traffic oblivious multi-path deterministic routing

Besides single-path deterministic routing, other routing schemes such as multi-path routing deterministic routing [12], [17] have also been developed for folded-Clos networks. InfiniBand allows multiple paths to be set-up between two endpoints [8]. In multi-path deterministic routing, multiple paths are established for each source-destination pair; and packets for one SD pair are distributed among the paths either in a deterministic or random manner. Single-path deterministic routing is a form of multi-path deterministic routing. Although multi-path routing schemes can achieve better load balance in folded-Clos networks than single-path routing [17], splitting packets from one SD pairs to multiple paths in a deterministic or random manner does not improve the nonblocking condition: under such conditions, the timing for a path to be used to route a packet in a SD pair is unpredictable (depending on the packet arrival timing and the routing scheme); to achieve nonblocking communication for any permutation, Lemma 1 still needs to hold, which leads to the same bound in Theorem 2 ($m \geq n^2$) for traffic oblivious multi-path deterministic routing schemes. As will be shown in the next section, when the routes for a SD pair can be adapted based on the traffic pattern, the nonblocking condition can be improved.

V. ADAPTIVE ROUTING

As discussed earlier, global adaptive routing where routes may adapt based on the whole communication pattern is equivalent to routing with a centralized controller; and the nonblocking conditions for such schemes have been established. In this work, we consider *local* adaptive routing where routes may adapt based on the information that is available locally to each switch. By developing a local adaptive routing algorithm that uses less than n^2 upper level switches ($m < n^2$) in $ftree(n, m, r)$ to achieve nonblocking communication, we show that local adaptive routing improves the nonblocking condition over deterministic routing, which indicates that more cost-effective nonblocking $ftree(n, m, r)$ can be built with local adaptive routing.

For $ftree(n, m, r)$, routing adaptivity can only be achieved in input switches. Once a packet reaches the top level switch, there is only one path to each destination and no adaptivity is possible. Hence, we will focus on local adaptive routing algorithms where the routes adapt only based on the local traffic pattern in each input switch. Our proposed local adaptive routing algorithm has the following properties.

- For a given communication pattern, the algorithm assigns one path to carry all traffics for one SD pair. Routing adaptivity is reflected in the fact that the paths for the same SD pair in different patterns may be different.
- For a given communication pattern, the path for a SD pair is determined based on the SD pairs in the communication pattern whose sources are in the same switch (adapt based on local information). SD pairs from different switches are routed independently.

By routing SD pairs from different switches independently, local adaptive routing algorithms with the above two properties, including our proposed routing algorithm, can be realized in a distributed manner by implementing the routing logic in each of the input switches, which have the information of all SD pairs from the switches. In the distributed implementation, the algorithm does not require global information to be shared among different switches. Each input switch adapts based on its local traffic pattern. Routing and traffic patterns in other switches do not affect the routing decision. Note that the terms *SD pairs from different switches* and *SD pairs from the same switches* are defined in Section III.

In the rest of the paper, the term *local adaptive routing algorithm* will refer to local adaptive routing algorithms with these two properties. The following lemma defines a class of local adaptive routing algorithms that are the building blocks of our proposed scheme.

Lemma 3: Let (s_1, d_1) and (s_2, d_2) be two arbitrary SD pairs in a communication pattern where $d_1 \neq d_2$ are in the same switch. If a local adaptive routing algorithm for $ftree(n, m, r)$ guarantees to route such two SD pairs through different top level switches, then routes for any two SD pairs in any permutation whose sources are in different input switches will not have network contention using the local adaptive routing algorithm.

Proof: Let us denote $SRC(s, d)$ the source switch of SD pair (s, d) and $DST(s, d)$ the destination switch of SD pair (s, d) . Let (s_1, d_1) and (s_2, d_2) be arbitrary two SD pairs in a permutation where s_1 and s_2 are in different switches ($SRC(s_1, d_1) \neq SRC(s_2, d_2)$). We will prove that under the assumption in the lemma, these two SD pairs will not have contention. Let (s_1, d_1) be routed through top level switch A and (s_2, d_2) be routed through top level switch B (A and B may be the same). Since $SRC(s_1, d_1) \neq SRC(s_2, d_2)$, uplinks $SRC(s_1, d_1) \rightarrow A$ and $SRC(s_2, d_2) \rightarrow B$ are different regardless whether $A = B$ or not, and there is no contention in the uplinks for the two SD pairs. For the downlink $A \rightarrow DST(s_1, d_1)$ and $B \rightarrow DST(s_2, d_2)$, there are two cases. When $DST(s_1, d_1) = DST(s_2, d_2)$, since (s_1, d_1) and (s_2, d_2) are from one permutation communication,

$d_1 \neq d_2$. By the assumption of this lemma, we have $A \neq B$ and thus, the downlinks for the two SD pairs are different and there is no network contention. When $DST(s_1, d_1) \neq DST(s_2, d_2)$, regardless whether $A = B$ or not, the downlinks $A \rightarrow DST(s_1, d_1)$ and $B \rightarrow DST(s_2, d_2)$ are different and there is no contention. \square

Lemma 3 states that for a class of local adaptive routing algorithms that guarantee to use different top level switches to route SD pairs with different destinations in the same switch, SD pairs from different switches in a permutation will not have contention. We will use the term **Class DIFF** to denote this class of local adaptive routing algorithms. Lemma 3 implies the following.

- 1) Using a Class DIFF algorithm, SD pairs from different switches can be routed independently: paths for SD pairs from different switches will not have contention.
- 2) A Class DIFF algorithm that also avoids contention for SD pairs from the same switch will achieve nonblocking communication. Hence, to design a routing algorithm that achieves nonblocking communication using Class DIFF algorithms as building blocks, we can focus on contention free routing for SD pairs from the same switch.

For any $f_{tree}(n, m, r)$, there exists a constant c such that $r \leq n^c$. For practical folded-Clos networks, c is a small constant. For example, in $f_{tree}(n, m, r = n^2)$, $c = 2$. In $f_{tree}(n, m, r = n^2 + n)$, $c = 3$. In our adaptive routing algorithm, we number the r bottom level switches with c n -based digits: $s_{c-1}s_{c-2}\dots s_0$, $0 \leq s_i \leq n-1$ for all $0 \leq i \leq c-1$; and we number the $r \times n$ leaf nodes (sources and destinations) with $c+1$ n -based digits: $s_{c-1}s_{c-2}\dots s_0p$, where $s_{c-1}s_{c-2}\dots s_0$ is the switch that the leaf node is in, and $0 \leq p \leq n-1$ is the local node number within switch $s_{c-1}s_{c-2}\dots s_0$. For a node $s_{c-1}s_{c-2}\dots s_0p$, we will say that p is the first digit of the node number, s_0 is the second digit, ..., s_{c-1} is the $c+1$ -th digit of the node number.

Our adaptive routing algorithm for $f_{tree}(n, m, r)$ uses Class DIFF schemes and schedules SD pairs from each switch independently. SD pairs from each switch are routed in phases. In each phase, SD pairs are routed over $(c+1) \times n$ top level switches. We will use the term **configuration** to denote the group of $(c+1) \times n$ top level switches used in one scheduling phase. The same routing logic is applied for all configurations until all SD pairs in a pattern are routed. Within each configuration, we further group the $(c+1) \times n$ top level switches into $c+1$ sets of n switches. We will call each set of n top level switches, a *partition*. In each of the partitions, the n top level switches are numbered from 0 to $n-1$.

Let $s_{c-1}s_{c-2}\dots s_0p$ be a generic destination node number. In our adaptive routing algorithm, the k -th partition ($1 \leq k \leq c+1$) in the $c+1$ partitions in a configuration, is used to route traffics to destinations with different k -th digits. Specifically, in the first partition, top level switch i , $0 \leq i \leq n-1$ is only used to carry SD pairs with destinations whose local node number $p = i$ for all bottom level switches: only SD pairs whose destinations have different

local node numbers are routed through this partition. Similarly, the second partition is used to route SD pairs to destinations with different second digits. Specifically, in the second partition, top level switch i carries traffics (from any sources) to destinations $(s_{c-1}\dots s_1(s_0 = i)(p = 0))$, $(s_{c-1}\dots s_1(s_0 = (i+1)\%n)(p = 1))$, ..., $((s_{c-1}\dots s_1(s_0 = (i+j)\%n)(p = j))$, ..., $((s_{c-1}\dots s_1(s_0 = (i+n-1)\%n)(p = n-1))$. Here, $\%$ is the modulo operation. The routing in other $c-1$ partitions is similar to that in the second partition. The i -th partition, $2 \leq i \leq c+1$, is used to route SD pairs to destinations with different i -th digit values (different s_{i-2} values). Specifically, in the i -th partition, top level switch i carries traffics to destinations $(s_{c-1}\dots (s_{i-2} = i)\dots s_0(p = 0))$, $(s_{c-1}\dots (s_{i-2} = (i+1)\%n)\dots s_0(p = 1))$, ..., $((s_{c-1}\dots (s_{i-2} = (i+j)\%n)\dots s_0(p = j))$, ..., $((s_{c-1}\dots (s_{i-2} = (i+n-1)\%n)\dots s_0(p = n-1))$. Note that the routing applies to all SD pairs from all source switches.

Lemma 4: The routing in each partition belongs to Class DIFF.

Proof: For the first partition, any two destinations with node numbers $s_{c-1}\dots s_0(p = n_1)$ and $s_{c-1}\dots s_0(p = n_2)$ in the same switch $s_{c-1}\dots s_0$ will be routed through two different top level switches n_1 and n_2 in this partition. The routing belongs to Class DIFF by definition from Lemma 3. For the second partition, different destinations in the same switch are also routed through different top level switches. For an arbitrary bottom level switch $s_{c-1}\dots s_0$, SD pairs with destination $s_{c-1}\dots s_00$ are routed through top level switch s_0 , destination $s_{c-1}\dots s_01$ through top level switch $(s_0 - 1)\%n$, ..., and destination $s_{c-1}\dots s_0j$, $0 \leq j \leq n-1$, through top level switch $(s_0 - j)\%n$ in the partition. Thus, the routing for the second partition is also a Class DIFF routing algorithm. Following a similar logic for the routing in the second partition, the routing for the i -th partition, $2 \leq i \leq c+1$, is a Class DIFF routing algorithm. \square

As discussed earlier, using Class DIFF algorithms as building blocks in a routing algorithm, if one can guarantee that SD pairs from the same switch in any permutation can be routed without contention, the algorithm achieves nonblocking communication. The following lemma gives the condition for a set of SD pairs to be routed through each partition without causing contention.

Lemma 5: Let $(s_1, d_1 = s_{c-1}^1\dots s_0^1p^1)$, $(s_2, d_2 = s_{c-1}^2\dots s_0^2p^2)$, ..., $(s_k, d_k = s_{c-1}^k\dots s_0^k p^k)$ be a set of SD pairs from the same switch in a permutation. If d_1, d_2, \dots, d_k each has a different first digit p^j , $1 \leq j \leq k$, then all of the k SD pairs can be routed through the first partition without contention. If $(s_i^j - p^j)\%n$, $1 \leq j \leq k$, are different, then all of the k SD pairs can be routed through the $i+2$ -th partition without contention.

Proof: Since the routing in the partition is a Class DIFF routing algorithm as discussed earlier, from Lemma 3, routes for the set of SD pairs will have no contention with paths for SD pairs from other switches in each partition.

Now consider the routes for SD pairs from the same source. When d_1, d_2, \dots, d_k each has a different first digit p^j , $1 \leq j \leq k$, then all of the k SD pairs will be routed through different switches p^j , $1 \leq j \leq k$, in the first partition. The paths for all

of the SD pairs are link-disjoint. Similarly, when $(s_i^j - p^j) \% n$, $1 \leq j \leq k$, are different, each of the k SD pairs will be routed through a different top level switch $(s_i^j - p^j) \% n$ in the $i + 2$ -th partition: the paths for all the SD pairs are link-disjoint. Thus, All of the k SD pairs can be routed through the $i + 2$ -th partition without contention. \square

Lemma 5 shows that each of the partitions in a configuration can be used to route a set of SD pairs that satisfy the conditions. In the worst case, at least one SD pair from each switch can be routed through each partition without causing contention. We will call a set of SD pairs that can be routed through a partition without causing contention *the set of SD pairs that can be routed through the partition*.

Using the routing logics, our adaptive routing algorithm for nonblocking $f_{tree}(n, m, r)$, NONBLOCKINGADAPTIVE, is described in Fig. 4. Since the routing in each partition is a Class DIFF routing algorithm, routing SD pairs from different switches for the same partition will not have contention and can be done independently. The routing algorithm routes SD pairs from each source switch independently. It considers configurations one at a time, greedily finds the largest number of SD pairs that can be routed through one of the unused partitions in each configuration, and routes the SD pairs to partition. This process is repeated with more configurations until all SD pairs are routed. After the configurations for SD pairs from all switches are computed, the algorithm merges the routes for SD pairs from different source switches (lines (14) and (15)): the corresponding partitions in each configuration for SD pairs from different switches can be routed through the same n top level switches without contention from Lemma 4. In the description, we assume that there are sufficient number of top level switches to be allocated. In the following, we will first prove this algorithm achieves nonblocking communication for any permutation and then give the upper bound of the number of top level switches required for this algorithm in Theorem 5.

Theorem 4: Algorithm NONBLOCKINGADAPTIVE results in nonblocking communication.

Proof: From Lines (7) and (8) in Fig 4, we can see that the NONBLOCKINGADAPTIVE only routes SD pairs that can be routed on a partition to the partition. It follows that the communication is nonblocking for any permutation (Lemma 5). \square

Analyzing the number of top level switches needed by the algorithm is more challenging. Before we prove the upper bound for the number of top level switches needed in NONBLOCKINGADAPTIVE, we will show that this algorithm requires less than n^2 top level switches for any permutation in $f_{tree}(n, m, r)$. For any two SD pairs from a switch in a permutation, the destinations are different: they differ in at least one digit in the $c + 1$ n -based digits representation. The two SD pairs can be routed through one partition in a configuration: if the first digits of the destinations are different, the two SD pairs can be routed through the first partition; if the first digits are the same, there exists an j such that the j -th digits of the destinations are different, and the two SD

Algorithm NONBLOCKINGADAPTIVE:

Input: A permutation P

Output: the routes for all SD pairs in P

- (1) Let P^i , $0 \leq i \leq r - 1$, be the set of SD pairs in P from switch i ;
- (2) For each P^i , $0 \leq i \leq r - 1$, do
- (3) $x_i = 0$;
- (4) While (P^i is not empty) do
- (5) $C_{x_i}^i =$ new configuration; x_i++ ;
- (6) While ($(P^i$ is not empty) and (C_x^i has unused partitions)) do
- (7) Find the largest subset of P^i , $LSET$, that can be routed on one of the unused partition, $PART$, in $C_{x_i}^i$;
- (8) Route SD pairs in $LSET$ on $PART$;
- (9) Mark $PART$ as used;
- (10) $P^i = P^i - LSET$;
- (11) End while
- (12) End while
- (13) End for
- (14) Let $totalconf$ be the largest x_i , $0 \leq i \leq r - 1$;
- (15) For $j=0$ to $totalconf - 1$ do
- merge corresponding partitions in C_j^i , $0 \leq i \leq r - 1$;
- (16) End for

Fig. 4. An adaptive routing algorithm for nonblocking $f_{tree}(n, m, r)$

pairs can be routed through the j -th partition ($s_{j-2} - p$'s are different). Hence, the $LSET$ found in line (7) in the first iteration after a new configuration is allocated will have at least two SD pairs when there are more than one SD pair in P^i . In other iterations, when P^i is not empty, the $LSET$ at least contains one SD pair from each switch since each unused partition can at least route one SD pair for each switch with no contention. Hence, the $c + 1$ partitions in one configuration can route at least route $c + 2$ SD pairs for each source switch. Since each switch can have at most n SD pairs in a permutation, at most $\frac{n}{c+2}$ configurations and $\frac{n}{c+2} \times (c + 1) \times n = \frac{c+1}{c+2}n^2$ top level switches are needed, which is less than the n^2 switches required with deterministic routing. In the following, we will show that NONBLOCKINGADAPTIVE improve the nonblocking condition asymptotically.

Lemma 6: Consider a set of numbers encoded with $c + 1$ n -based digits, $d_c d_{c-1} \dots d_0$. For any k different numbers, there exists at least one i , $0 \leq i \leq c$, such that there are at least $k^{\frac{1}{2(c+1)}}$ numbers in the set of k different numbers either with different d_0 or with different $(d_i - d_0) \% n$'s.

Proof: Let the number of different values of each digit d_i among the k different numbers be X_i . If $X_i < k^{\frac{1}{c+1}}$ for all $0 \leq i \leq c$, then at most $X_0 \times X_1 \times \dots \times X_c$ different numbers can be in the set. Hence, $k \leq X_0 \times X_1 \times \dots \times X_c < k$, which cannot be true. Hence, there exists at least one i such that $X_i \geq k^{\frac{1}{c+1}}$.

If $i = 0$, there exists $k^{\frac{1}{c+1}}$ (more than $k^{\frac{1}{2(c+1)}}$) numbers

with different d_0 and the lemma is proven. If $i \neq 0$, there exists a set of $k^{\frac{1}{c+1}}$ numbers with different d_i 's. If this case, if the number of different d_0 's in the set of numbers is less than $k^{\frac{1}{2(c+1)}}$, the number of different $(d_i - d_0) \% n$'s in the set is at least $k^{\frac{1}{2(c+1)}}$. \square

Theorem 5: Let $r \leq n^c$, where c is a constant. Algorithm NONBLOCKINGADAPTIVE requires at most $O(n^{2 - \frac{1}{2(c+1)}})$ top level switches to route any permutation in $ftree(n, m, r)$. *Proof:* Consider the number of configurations needed by NONBLOCKINGADAPTIVE for any permutation communication. Each source switch will have at most n SD pairs to route in a permutation. Let $s_{c-1} \dots s_0 p$ be a generic destination in the permutation. From Lemma 6, among the n different destinations in the n SD pairs, which are represented by $c+1$ n -based digits, there exists an i , $0 \leq i \leq c$, such that there are $n^{\frac{1}{2(c+1)}}$ destinations with different first digits or with different $(s_{i-2} - p) \% n$'s. This set of $n^{\frac{1}{2(c+1)}}$ SD pairs with one for each of those destinations can be routed through a partition (Lemma 5). Since the largest subset of P^i that can be routed on one partition is found in line (7) in Fig. 4, the first partition after each new configuration is allocated will route at least $n^{\frac{1}{2(c+1)}}$ SD pairs for each source switch when $|P^i| = n$. Let $T(n)$ be the number of configurations needed when each switch has n SD pairs in a permutation to be routed. Note that since all switches route SD pairs independently and share the same set of upper level switches, the entire permutation P will also require $T(n)$ configurations to be routed. We have $T(n) \leq T(n - n^{\frac{1}{2(c+1)}}) + 1$.

For $n > X > \frac{n}{2}$, $n^{\frac{1}{2(c+1)}} > X^{\frac{1}{2(c+1)}} > (\frac{n}{2})^{\frac{1}{2(c+1)}}$. Hence,

$$\begin{aligned} T(n) &\leq T(n/2) + \frac{\frac{n}{2}}{(\frac{n}{2})^{\frac{1}{2(c+1)}}} \\ &= T(n/2) + (\frac{n}{2})^{1 - \frac{1}{2(c+1)}} \\ &\leq (\frac{n}{2})^{1 - \frac{1}{2(c+1)}} (1 + (\frac{1}{2})^{1 - \frac{1}{2(c+1)}} + (\frac{1}{2})^{2 \times (1 - \frac{1}{2(c+1)})} + \dots) \\ &= O(n^{1 - \frac{1}{2(c+1)}}) \end{aligned}$$

Hence, the number of top level switches needed for the algorithm is not more than $T(n) \times (c+1) \times n = O(n^{2 - \frac{1}{2(c+1)}})$, where c is a constant. \square

Algorithm NONBLOCKINGADAPTIVE asymptotically improves the nonblocking condition for $ftree(n, m, r)$ with deterministic routing. However, it may not achieve the lower bound of m for local adaptive routing. The tight nonblocking condition for $ftree(n, m, r)$ with local adaptive routing is, thus, still open for investigation.

VI. CONCLUSION

We study folded-Clos networks that are nonblocking in computer communication environments and develop techniques to construct such networks. We show that it is not effective to build nonblocking folded-Clos with small top level switches. We prove that for $ftree(n, m, r)$ with single-path deterministic routing, the nonblocking condition is $m \geq n^2$

when $r \geq 2n + 1$. We give the single-path routing scheme that can be used to build nonblocking $ftree(n, m = n^2, r)$, which is optimal. We further prove that using local adaptive routing, the nonblocking condition can be improved over deterministic routing. The results indicate that it is possible to build large nonblocking folded-Clos networks in computer communication environments using smaller components. This paper leaves an open problem to be investigated in the future: what is the tight nonblocking condition for $ftree(n, m, r)$ with local adaptive routing?

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